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THE DEPTH OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS

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Abstract. We show in ZFC, that the depth of ultraproducts of Boolean Algebras may be bigger than the ultraproduct of the depth of those Boolean Algebras.

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§0 Introduction

Monk has looked systematically at cardinal invariants of Boolean Algebras. In particular, he has looked at the relations between $\text{inv}(\prod\limits_{i<\kappa} \mathbf{B}_i/D)$ and $\prod\limits_{i<\kappa} \text{inv}(\mathbf{B}_i)/D$, i.e., the invariant of the ultraproducts of a sequence of Boolean Algebras vis the ultraproducts of the sequence of the invariants of those Boolean Algebras for various cardinal invariants inv of Boolean Algebras. That is: is it always true that $\text{inv}(\prod\limits_{i<\kappa} \mathbf{B}_i/D) \leq\ \text{inv}(\prod\limits_{i<\kappa} \mathbf{B}_i/D)$? is it consistently always true? Is it always true that $\prod\limits_{i<\kappa} \text{inv}(\mathbf{B}_i)/D \leq\ \text{inv}(\Pi \mathbf{B}_i/D)$? is it consistenly always true? See more on this in Monk [Mo96]. Roslanowski Shelah [RoSh 534] deals with specific inv and with more on kinds of cardinal invariants and their relationship with ultraproducts. Monk [Mo90a], [Mo96], in his list of open problems raises the question for the central cardinal invariants, most of them have been solved by now; see Magidor Shelah [MgSh 433], Peterson [Pe97], Shelah [Sh 345], [Sh 462], [Sh 479], [Sh 589, §4], [Sh 620], [Sh 641], [Sh 703], Shelah and Spinas [ShSi 677].

We here solve problem 12 of [Mo96], pg.287 in ZFC constructing an example. This example works for the length too. As in several earlier cases we use pcf theory to resolve the question near singular cardinals, see [Sh:g].

§1 On problem 12,p.287 of [Mo96],Monk 1996 book

1.1 Claim. *1) Assume*

(a) $\mu = \mu^\kappa > 2^\kappa$
(b) μ *singular*, $\mathrm{cf}(\mu) = \theta$.

Then there are Boolean Algebras B_i for $i < \kappa$ such that

(α) $\mathrm{Depth}(\mathbf{B}_i) \le \mu$ *for* $i < \kappa$, *hence*
(α)′ *for any ultrafilter D on* $\kappa, \mu = \prod\limits_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$
(β) *for any uniform ultrafilter D on κ, the Boolean Algebra* $\prod\limits_{i<\kappa} B_i/D$ *has depth* $\ge \mu^+$.

2) We can replace in $(\alpha) + (\beta)$, μ by μ_1 if $\mu_1 < \mathrm{pp}(\mu)$ except in very rare cases, in particular it suffices to assume that $\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}}(\mu) > \mu_1$ or it suffices then $\lambda = \mathrm{tcf}(\prod a, <_J), J$ *an ideal of* $\mathfrak{a}, \emptyset = \bigcap\limits_{i<\kappa} \mathfrak{b}_i, \mathfrak{b}_i \in J$ *decreasing with empty intersections* $\theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\mathfrak{a} \cap \theta) < \lambda$.

Proof of 1.1. This is a special case of 1.3.

1.2 Remark. Clearly for any given κ there are many such μ's, e.g., $\beth_{\kappa^+}$.

1.3 Claim. *Assume*

(a) *J is an ideal on* $\mathfrak{a}$, $\sup(\mathfrak{a}) = \mu, \mu$ *singular and* $\mu = \lim_J(\mathfrak{a})$; *that is* $(\forall \mu_1 < \mu)[\mathfrak{a} \cap \mu_1 \in J)$ *and* $\theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\mathfrak{a} \cap \beta) < \mu$
(b) $\lambda = \mathrm{tcf}(\prod \mathfrak{a}, \le_J)$ *as witnessed by* $\langle f_\alpha : \alpha < \lambda\rangle$
(c) $\mathfrak{b}_i \in J^+$ *for* $i < \kappa, \mathfrak{b}_i$ *decreasing with i and* $\emptyset = \cap\{\mathfrak{b}_i : i < \kappa\}$
(d) *D is a uniform ultrafilter on κ.*

Then for some sequence $\langle \mathbf{B}_i : i < \kappa\rangle$ of Boolean Algebras we have:

(α) $\mathrm{Depth}^+(\prod\limits_{i<\kappa} \mathbf{B}_i/D) > \lambda$ *(if $\lambda = \mu^+$ this means* $\mathrm{Depth}(\prod\limits_{i<\kappa} \mathbf{B}_i/D) \ge \lambda$)
(β) $\mathrm{Depth}^+(\mathbf{B}_i) \le \lambda$ *(if $\lambda = \mu^+$ this means* $\mathrm{Depth}(\mathbf{B}_i/D) \le \mu$).

Proof. We can find $\langle f_\alpha : \alpha < \lambda\rangle$ which is $<_J$-increasing cofinal in $(\pi\mathfrak{a}, <_J)$ and satisfies $\theta \in \mathfrak{a} \Rightarrow |\{f_\alpha \restriction (\mathfrak{a} \cap \theta) : \alpha < \lambda\}| < \theta$ (see [Sh:g, II,3.5,pg.65]). We define a function $\theta : [\lambda]^2 \to \kappa$ by: for $\alpha \neq \beta < \lambda$ we let $\theta\{\alpha,\beta\} = \mathrm{Min}\{\theta \in \mathfrak{a} : f_\alpha(\theta) \neq f_\beta(\theta)\}$ and we define a two place relation $<_i$ on λ by: $\alpha <_i \beta$ iff $\theta\{\alpha,\beta\} \in \mathfrak{a}\backslash\mathfrak{b}_i$ & $f_\alpha(\theta\{\alpha,\beta\}) < f_\beta(\theta\{\alpha,\beta\})$. Now

$\circledast_1$ $\leq_i$ is a partial order of λ.
[Why? Assume $\alpha <_i \beta <_i \gamma$.

Now

Case 1: $\alpha = \beta \vee \beta = \gamma$: trivial.

Case 2: $\theta\{\alpha,\beta\} < \theta\{\beta,\gamma\}$ so

$$\theta\{\alpha,\gamma\} = \theta\{\alpha,\beta\},$$

$$\begin{aligned}(f_\alpha(\theta\{\alpha,\beta\}), f_\beta(\theta\{\alpha,\beta\}) &= (f_\alpha(\theta\{\alpha,\gamma\}), f_\beta(\theta\{\alpha,\gamma\}) \\ &= (f_\alpha(\theta\{\alpha,\gamma\}), f_\gamma(\theta\{\alpha,\gamma\})\end{aligned}$$

and we are done.

Case 3: $\theta\{\alpha,\beta\} > \theta\{\beta,\gamma\})$.
Similarly.

Case 4: $\theta(\alpha,\beta) = \theta(\beta,\gamma)$. Call it θ. So $f_\alpha \restriction \theta = f_\beta \restriction \theta = f_\gamma \restriction \theta$ and $f_\alpha(\theta) < f_\beta(\theta) < f_\gamma(\theta)$, hence $\theta\{\alpha,\gamma\} = \theta$ and $f_\alpha <_i f_\gamma$ as required. So $\circledast_1$ holds.]

Let $\mathbf{B}_i = BA[(\lambda, <_i)]$ where for a partial order $(I, \leq_I)$, $BA[(I, \leq_I)]$ is the Boolean Algebra generated by $\{x_t : t < I\}$ freely except that

$\circledast_2$ $x_s \leq x_t$ when $s \leq_I x_t$.

Now

$\circledast_3$ in $\mathbf{B} = \prod_{i<\kappa} \mathbf{B}_i/D$, there is an increasing sequence of length λ.
[Why? Let $a_\alpha = \langle x_\alpha : \alpha < \kappa\rangle/D$, now if $\alpha < \beta$ then $\theta\{\alpha,\beta\} \notin \mathfrak{b}_i \Rightarrow B_i \models$ "$x_\alpha < x_\beta$" and $\alpha < \beta \Rightarrow \{i < \kappa : \theta(\theta,\beta) \notin \mathfrak{b}_i\} \in D$ as D is a uniform ultrafilter on κ and the sequence $\langle \mathfrak{b}_i : i < \kappa\rangle$ decreases with intersection $\emptyset$ we are done easily. Together $\alpha < \beta \Rightarrow B \models a_\alpha < a_\beta$; so $\langle a_\alpha ; \alpha < \lambda\rangle$ is as required so $\circledast_3$ holds.]

So it is enough to prove (as done in the rest of the proof).

$\circledast_4$ $\text{Depth}^+(\mathbf{B}_i) \le \lambda$.

Toward contradiction, assume $\langle a_\alpha : \alpha < \lambda\rangle$ is an $<_{B_i}$-increasing sequence of members of $\mathbf{B}_i$. Let $a_\alpha = \sigma_\alpha(x_{\gamma(\alpha,0)}, \dots, x_{(\alpha,n_\alpha-1)})$ where σ_α is a Boolean term and

$$\gamma(\alpha,0) < \gamma(\alpha,1) < \ldots < \gamma(\alpha,n_\alpha - 1) < \lambda.$$

Without loss of generality $\sigma_\alpha = \sigma_*$ so $n_\alpha = n_*$ and $\theta\{\gamma(\alpha,\ell_1),\gamma(\alpha,\ell_2)\}$ is the same for all $\alpha < \lambda$, say is θ_{ℓ_1,ℓ_2}. Now without loss of generality for some $\theta_* \in \mathfrak{a}$ satisfying $\ell_1 < \ell_2 < n_* \Rightarrow \theta_{\ell_1,\ell_2} < \theta_*$ we have $\ell < n_*$ & $\alpha < \beta < \lambda \Rightarrow f_{\gamma(\alpha,\ell)} \restriction (\mathfrak{a} \cap \theta_*) = f_{\gamma(\beta,\ell)} \restriction (\mathfrak{a} \cap \theta_*)$.
[Why? Recall that $\theta \in \mathfrak{a} \Rightarrow \theta > |\{f_\alpha \restriction \theta : \alpha < \lambda\}|$ and $\theta \in \mathfrak{a} \Rightarrow \theta < \lambda = \text{cf}(\lambda)$.]

Also without loss of generality for some $m_* < n_*, \ell < m_* \Rightarrow \gamma(\alpha,\ell) = \gamma_*(\ell)$ and $\alpha < \beta \Rightarrow \gamma(\alpha,n_*-1) < \gamma(\beta,m_*)$. By [Sh:g, II,4.10A,4.10B,pg.76,77] as $\mathfrak{b}_i \in J^+$ we can find $\theta^* \in \mathfrak{b}_i \backslash \theta_*$ and $\alpha < \beta$ such that

$\boxtimes^\theta_{\alpha,\beta}$ $\theta^* = \theta\{\gamma(\alpha,\ell_1),\gamma(\beta,\ell_2)\}$ whenever $\ell_1 \ne \ell_2 \in \{m_*,\dots,n_*-1\}$.

Now let $I = \{\gamma(\alpha,\ell),\gamma(\beta,\ell) : \ell < n_*\}$. Now we know that $BA[(I, \le_i \restriction I)]$ is a Boolean subalgebra of B_i hence $BA[(I,\le_i\restriction I)] \models$ "$\sigma_*(\dots,x_{\gamma(\alpha,\ell)},\dots) < \sigma_*(\dots,x_{\gamma(\beta,\ell)},\dots)$". But every automorphism π of $(I,\le_i\restriction I)$ induces an automorphism $\hat\pi$ of $BA[(I,\le_i\restriction I)]$, but the permutation π interchanging $\gamma(\alpha,\ell)$ with $\gamma(\beta,\ell)$ is an automorphism of $(I,\le_i\restriction I)$ so $BA[(I,\le_i\restriction I)] \models$ "$\hat\pi(\sigma_x(\dots,x_{\gamma(\alpha,\ell)},\dots) < \hat\pi(\sigma_*(\dots,x_{\gamma(\beta,\ell)},\dots))$" but this gives a contradiction. $\square_{1.3}$

1.4 Claim. *In 1.3 we can add*

$(\alpha)''$ *Length*$^+(\mathbf{B}_i) \le \lambda$ *for* $i < \kappa$.

Proof. The same proof works, only concerning $\circledast_4$, it is now

$\circledast_i$ $\text{Length}(\mathbf{B}_i) \le \lambda$.

The proof is the same but we do not know that $\text{BA}[(I,\le_i\restriction I)] \models$ "$\sigma_*(\dots,x_{\gamma(\alpha,\ell)},\dots)_{\ell<n_*} < \sigma_*(\dots,x_{\gamma(\beta,\ell)},\dots)_{\ell<n_*}$" but only know that $\text{BA}[(I,\le_i\restriction I)] \models$ "the elements $\sigma_*(\dots,x_{\gamma(\alpha,\ell)},\dots)_{\ell<n_*}$ and $\sigma_*(\dots,x_{\gamma)\beta,\ell)},\dots)_{\ell<n_*}$ are comparable. $\square_{1.4}$

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